

Boundary energy and boundary states in integrable quantum field theories

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Abstract: We study the ground state energy of integrable 1+1 quantum field theories with boundaries (the genuine Casimir effect). In the scalar case, this is done by introducing a new, “R-channel TBA”, where the boundary is represented by a boundary state, and the thermodynamics involves evaluating scalar products of boundary states with all the states of the theory. In the non-scalar, sine-Gordon case, this is done by generalizing the method of Destri and De Vega. The two approaches are compared. Miscellaneous other results are obtained, in particular formulas for the overall normalization and scalar products of boundary states, exact partition functions for the critical Ising model in a boundary magnetic field, and also results for the energy, excited states and boundary S-matrix of $O(n)$ and minimal models.

1. Introduction

The main purpose of this paper is to study the ground state energy of 1+1 integrable relativistic quantum field theories with boundaries. This involves several questions. One is the energy associated with a boundary for an infinite system, the other is the way the energy of the theory on an interval varies with its length - the “genuine” Casimir effect.

Some of these properties are most easily studied directly in a continuum formalism, using a scattering description for either a massive or massless bulk theory, the information on the boundary being encoded into a reflection matrix. Other properties are most easily studied using a lattice regularization.

In computing energies, we shall use detailed information about the boundary states, and this work also presents new results on that question. For instance, the “R channel” of the Thermodynamical Bethe Ansatz (TBA) approach provides expressions for the scalar products of boundary states.

For a Quantum Field Theory defined on a torus, the standard way to compute its ground state energy is through the Thermodynamic Bethe Ansatz [1]. If the theory is defined on a circle of circumference R , one switches to a modular transformed point of view where now the theory is defined on a circle of very long circumference L and at temperature $1/R$. The free energy F of the theory in the “ R channel” can be computed using TBA, and it is simply related to the ground state energy $E^0(R)$ of the theory in the “ L channel” by $F = -TLE^0(R)$.

Consider now a quantum field theory on a cylinder of finite length R and circumference L , with some boundary conditions (a, b) at the ends of the cylinder. There are two possible ways of viewing the partition function: in the ‘R-channel’ where time evolution is in the R direction, and in the ‘L-channel’ where time evolution is in the L direction. (See figure 1). This problem is more difficult because the theory looks rather different in the two channels. To use the same argument and do a TBA in the R channel, one now needs to control the effect of boundary states on the thermodynamics. This is discussed in section 2 for the simple case of the Ising model to clarify a few delicate points, and in section 3 for the general case of a scalar theory. The TBA in the L channel gives rise to a different kind of information, related to the normalization of boundary states. This is discussed in section 4. Finally, all these results can be generalized to the case where the theory is still

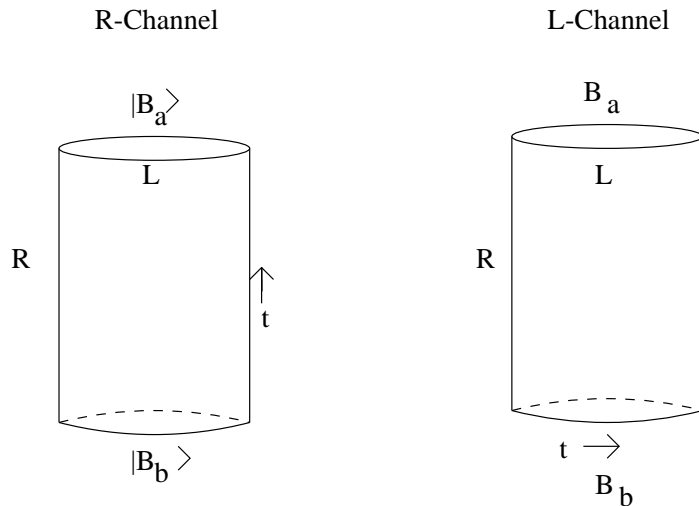


Figure 1. Cylindrical geometry in the R and L channels.

conformal in the bulk but has a boundary interaction that breaks the conformal invariance (section 5).

In the second part of this paper we address the same questions for a non scalar theory, namely the sine-Gordon model with Dirichlet boundary conditions. This case is technically more difficult. Instead of rederiving a TBA in the R channel, we have generalized the beautiful approach of Destri and de Vega [2] for a system with boundaries. This requires working for a while with the lattice theory, here chosen to be the XXZ model with boundary magnetic fields.

In section 6 we recall the various elements of the model, and carry out the TBA in the L channel. As a result, we are able to obtain the boundary ground state energy of the sine-Gordon model with Dirichlet boundary conditions. In section 7 we write the ground state energy. In section 8 we use this result for minimal models, proposing in particular boundary conditions that correspond to excited states of the theory. Section 9 collects conclusions. In particular we show that the DDV equations are actually essentially similar to the equations for a scalar theory derived in section 3.

Part I: Scalar theories

2. Ground state energy of the Ising model with boundaries

In the bulk, the Ising model can be described in terms of massive fermionic operators $A(\theta)$ and $A^\dagger(\theta)$ with the usual anti-commutation relations, $\{A(\theta), A^\dagger(\theta')\} = 2\pi\delta(\theta - \theta')$, where θ is the rapidity variable. The mass m of the fermionic field is proportional to $T_c - T$. The S -matrix in the bulk is simply -1 . In this section, we are interested in the explicit computation of the ground state energy of this model for different boundary conditions. Let us consider then the Ising model in the low temperature phase ($T < T_c$) in the geometry of figure 2, i.e. a long strip of horizontal length L and vertical width R . At the extremities of the horizontal axis, the order parameter of the model is subjected to periodic boundary conditions while in the vertical direction it satisfies boundary conditions of type a and b . Thus, we are effectively considering the partition function on a cylinder of circumference L and length R , with different boundary conditions on each end of the cylinder. (See figure 1.) As we discuss in more detail below, such boundary conditions are expressed in terms of boundary states $|B_a\rangle$ and $|B_b\rangle$.

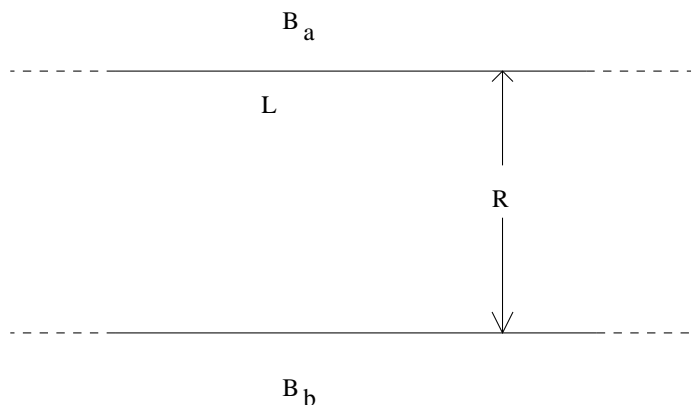


Figure 2. Strip geometry.

Depending of our quantization scheme, we have two possible ways to compute the partition function of the system. The first possibility consists in choosing as direction of time the horizontal axis and therefore the partition function will be expressed as

$$Z_{ab} = \text{Tr} e^{-LH_{ab}}, \quad (2.1)$$

where H_{ab} is the Hamiltonian relative to the system with boundary conditions (a, b) . In the second method the time evolution takes place along the vertical axis and therefore the partition function is given by the matrix element of the time evolution operator between the boundary states, i.e.

$$Z_{ab} = \langle B_a | e^{-RH} | B_b \rangle, \quad (2.2)$$

where now H is the Hamiltonian of the bulk system. The ground state energy $E_{ab}(R)$ is, by definition, the leading term arising in the large L limit of the first expression, eq. (2.1), i.e.

$$Z_{ab} \sim e^{-LE_{ab}^0(R)}. \quad (2.3)$$

However, in view of the equivalence of the two quantization schemes, we can compute this quantity by looking at the large L limit of the second expression, eq. (2.2).

Since eq. (2.2) employs the boundary states of the model, let us shortly recall their basic properties (for more detail see the original reference [3]). The Ising spins placed on the boundaries can be subjected to three possible boundary conditions, namely: (i) they can be frozen to one of the two possible fixed values \pm (fixed boundary condition); (ii) they can be completely free to fluctuate (free boundary condition); (iii) they can be coupled to a boundary magnetic field h (magnetic boundary condition). In the QFT description of the model, each of the above microscopic configurations corresponds to a boundary state $|B\rangle$, which for infinite length L , reads [3]

$$|B\rangle = g \exp \left[\int_0^\infty \frac{d\theta}{2\pi} K(\theta) A^\dagger(-\theta) A^\dagger(\theta) \right] |0\rangle. \quad (2.4)$$

Here g is an overall normalization which we shall discuss later. Since we are for the moment interested in the large L limit, we simply set this factor equal to unity. For simplicity, we ignore possible additional contributions to the boundary state from zero momentum particles. From the point of view of QFT, the boundary state can be therefore regarded as a particular state of the Hilbert space of the bulk theory, made of a superposition of pairs of particles of equal and opposite momentum (“Cooper pairs”). All information relative

to a particular boundary condition is encoded into the function $K(\theta)$ which can be seen as the elementary amplitude to create a virtual pair of particles.

For the Ising model at low temperature, the vacuum state $|0\rangle$ can be one of the two possible vacua of the model in this phase and the explicit expressions of the amplitude $K(\theta)$ for the three cases above considered are given by [3]

$$K_{free} = -i \coth \frac{\theta}{2}, \quad (2.5)$$

for the free boundary conditions,

$$K_{fixed} = i \tanh \frac{\theta}{2}, \quad (2.6)$$

for the fixed boundary conditions, and

$$K_h = i \tanh \frac{\theta}{2} \frac{k + \cosh \theta}{k - \cosh \theta}, \quad (2.7)$$

for magnetic boundary conditions, where $k = 1 - \frac{h^2}{2m}$. It is evident that by varying h , we can interpolate between the free boundary condition ($h = 0$) and the fixed one ($h \rightarrow \infty$).

After this brief discussion on the boundary states, let us come back now to the evaluation of eq. (2.2) in the limit $L \rightarrow \infty$. The first thing to consider is the action of the time evolution operator e^{-RH} on the boundary state $|B_b\rangle$. Since H is the Hamiltonian of the bulk theory, it is diagonalized on the basis of the multiparticle asymptotic states $|\theta_1, \dots, \theta_n\rangle$, with eigenvalues given by $m \sum_i^n \cosh \theta_i$. Expanding the boundary state $|B_b\rangle$ in terms of multiparticle states by using its definition (2.4), it is easy to see that the action of the time evolution operator on $|B_b\rangle$ simply reduces to a redefinition of the amplitude $K_b(\theta)$ of this state, namely

$$K_b(\theta) \rightarrow K_b(\theta, R) \equiv K_b(\theta) e^{-2mR \cosh \theta}. \quad (2.8)$$

Consequently, from an abstract point of view, the evaluation of the partition function Z_{ab} consists in computing the scalar product of two boundary states, one of them R dependent, i.e. $Z_{ab} = \langle B_a | B_b(R) \rangle$. To determine such a quantity, it is convenient to expand both boundary states in their multiparticle components. For the integrability of the theory, transition amplitudes between states with different number of particles are not allowed and therefore Z_{ab} can be expressed as

$$Z_{ab} = \sum_{N=0}^{\infty} \mathcal{Z}_{ab}^{(2N)}, \quad (2.9)$$

where

$$\begin{aligned} \mathcal{Z}_{ab}^{(2N)} &= \frac{1}{(N!)^2} \int_0^\infty \prod_{i,j=1}^N \frac{d\theta_i}{2\pi} \frac{d\theta'_j}{2\pi} \overline{K}_a(\theta_i) K_b(\theta'_j, R) \\ &< 0 | A(-\theta'_1) A(\theta'_1) \dots A(-\theta'_N) A(\theta'_N) A^\dagger(\theta_N) A^\dagger(-\theta_N) \dots A^\dagger(\theta_1) A^\dagger(-\theta_1) | 0 > \end{aligned} \quad (2.10)$$

The first term of this series is given by $Z_{ab}^{(0)} = < 0 | 0 > = 1$. The higher terms can be computed by using the Wick theorem. There is, however, a subtlety which already arises in the computation of the second term

$$\begin{aligned} \mathcal{Z}_{ab}^{(2)} &= \int_0^\infty \frac{d\theta}{2\pi} \int_0^\infty \frac{d\theta'}{2\pi} \overline{K}_a(\theta) K_b(\theta', R) (2\pi)^2 \delta(\theta - \theta') \delta(\theta - \theta') = \\ &= \delta(0) \int_0^\infty \frac{d\theta}{2\pi} \overline{K}_a(\theta) K_b(\theta) e^{-2mR \cosh \theta} \end{aligned} \quad (2.11)$$

The term $\delta(0)$ signals the divergence due to the infinite volume limit on which the boundary states are defined. To correctly extract the L dependence of this expression, we need to redefine the normalization of the multiparticle states, i.e. to provide a regularization of the δ function. If we were working in the momentum space, a possible regularization of the δ function is given by

$$\delta_L(p) = \frac{1}{2\pi} \int_{-\frac{L}{2}}^{\frac{L}{2}} dx e^{ipx} = \frac{1}{\pi p} \sin \frac{pL}{2}. \quad (2.12)$$

Using this definition, we can make sense of the square of the δ function and in the limit $L \rightarrow \infty$ we have $(\delta_L(p - q))^2 \rightarrow \frac{L}{2\pi} \delta(p - q)$. However, working in the rapidity variable, the final expression of the square of the δ function entering (2.11) acquires an extra term $m \cosh \theta$ due to the Jacobian of the transformation from the momentum to rapidity variable and the final result is to substitute

$$[\delta(\theta - \theta')]^2 \rightarrow \frac{mL}{2\pi} \cosh \theta \delta(\theta - \theta'). \quad (2.13)$$

This regularization allows us to keep track of the L dependence of our expressions and therefore to extract the ground state energy $E_{a,b}^0$. With this substitution, $Z_{ab}^{(2)}$ is now given by

$$Z_{ab}^{(2)} = mL \int_0^\infty \frac{d\theta}{2\pi} \cosh \theta \overline{K}_a(\theta) K_b(\theta) e^{-2mR \cosh \theta}. \quad (2.14)$$

For later convenience, it is useful to define the following quantities

$$I_N = \int_0^\infty \frac{d\theta}{2\pi} \cosh \theta \left[\overline{K}_a(\theta) K_b(\theta) e^{-2mR \cosh \theta} \right]^N. \quad (2.15)$$

With the new normalization of the states, the higher terms $\mathcal{Z}_{a,b}^{(2N)}$ in the series (2.9) are polynomials in the variable (mL) of order N , as explicitly shown by the first representatives

$$\begin{aligned} Z_{ab}^{(4)} &= \frac{(mL)^2}{2} I_1^2 - \frac{(mL)}{2} I_2, \\ Z_{ab}^{(6)} &= \frac{(mL)^3}{3!} I_1^3 - \frac{(mL)^2}{2} I_1 I_2 + \frac{(mL)}{3} I_3, \\ Z_{ab}^{(8)} &= \frac{(mL)^4}{4!} I_1^4 - (mL)^3 I_1^2 I_2 + \frac{(mL)^2}{2} \left[\frac{2}{3} I_1 I_3 + \left(\frac{I_2}{2} \right)^2 \right] - \frac{(mL)}{4} I_4, \end{aligned} \quad (2.16)$$

and so on. The resulting sum organizes into an exponential series

$$Z_{ab} = 1 + (mL)f(R) + \frac{(mL)^2}{2!} (f(R))^2 + \dots \frac{(mL)^n}{n!} (f(R))^n + \dots = \exp[mLf(R)] \quad (2.17)$$

where

$$f(R) = \sum_{N=1}^{\infty} (-1)^{N+1} \frac{I_N}{N} = \int_0^\infty \frac{d\theta}{2\pi} \cosh \theta \log \left(1 + \overline{K}_a(\theta) K_b(\theta) e^{-2mR \cosh \theta} \right). \quad (2.18)$$

Therefore, the ground state energy for boundary conditions (a, b) is given by

$$E_{ab}^0(R) = -\frac{m}{4\pi} \int_{-\infty}^{\infty} d\theta \cosh \theta \log \left(1 + \overline{K}_a(\theta) K_b(\theta) e^{-2mR \cosh \theta} \right). \quad (2.19)$$

The above result can also be easily derived as a special case of the general formula

$$\langle 0 | e^{(\tilde{a}, Ma)} e^{(a^\dagger, N\tilde{a}^\dagger)} | 0 \rangle = \det(1 + NM), \quad (2.20)$$

where $(\tilde{a}, Ma) = \sum_{n,m} \tilde{a}_n M_{nm} a_m$ and $\{a_n, a_m^\dagger\} = \{\tilde{a}_n, \tilde{a}_m^\dagger\} = \delta_{n,m}$, $\{\tilde{a}_n, a_m^\dagger\} = 0$. Specializing to our situation, the matrices M, N are kernels: $M(\theta, \theta') = \delta(\theta - \theta') \overline{K}(\theta)$, $N(\theta, \theta') = \delta(\theta - \theta') K(\theta) e^{-2mR \cosh \theta}$, and the determinant in (2.20) is a Fredholm determinant. Using $\log \det(1 + NM) = \text{tr} \log(1 + NM)$, and regulating the squares of delta functions as in (2.13), one obtains (2.19).

As a check of this expression, we can compare its ultraviolet limit $R \rightarrow 0$ with the result directly obtained by using CFT [4]. In the conformal limit, general expressions for

the partition function in the cylindrical geometry with boundaries were derived by Cardy [5]. In the R-channel, one has

$$Z_{ab} = \sum_i n_{ab}^i \chi_i(q), \quad (2.21)$$

where $q = \exp(-\pi L/R)$, χ_i are the characters, $\chi_i(q) = q^{-c/24} \text{Tr } q^{L_0}$, and n_{ab}^i denotes how many times the representation $[i]$ appears with the boundary conditions (a, b) . Thus, as L tends to infinity, one has

$$E_{ab}^0 = \frac{\pi}{R} \left(L_0^{(0)} - \frac{c}{24} \right), \quad (2.22)$$

where $L_0^{(0)}$ is lowest L_0 eigenvalue of the states in the representations $[i]$ for which $n_{ab}^i \neq 0$. Taking the limit $R \rightarrow 0$ in (2.19), the ground state energy assumes the scaling form

$$E_{ab}^0(R) \approx -\frac{1}{4\pi R} \int_0^\infty d\epsilon \log [1 + \lambda_{ab}(\infty) e^{-\epsilon}], \quad (2.23)$$

where

$$\lambda_{ab}(\theta) = \overline{K}_a(\theta) K_b(\theta). \quad (2.24)$$

Since $\lambda_{free,free}(\infty) = \lambda_{fixed,fixed}(\infty) = 1$,

$$E_{free,free}^0 = E_{fixed,fixed}^0 \approx -\frac{\pi}{48R} \quad (2.25)$$

With those boundary conditions, the ground state energy is dictated by the conformal family of the identity operator. For the mixed boundary conditions, we have instead $\lambda_{free,fixed}(\infty) = -1$ and therefore

$$E_{free,fixed}^0 \approx \frac{\pi}{24R}. \quad (2.26)$$

In this case, the ground state energy is ruled by the conformal family of the magnetization operator, of conformal dimension $\frac{1}{16}$. These results are in agreement with those obtained by Cardy [4].

3. The ground state energy of a scalar theory with boundaries and TBA in the R-channel

Reproducing the above computation in the case of a scalar interacting theory is more difficult. This is because the particles scatter non trivially, so care must be exercised in computing scalar products. Also, the allowed rapidities of the physical states are non

trivial solutions of Bethe type equations. The simplest way to proceed is to use a “thermal approach”, generalizing the standard Thermodynamic Bethe Ansatz (TBA).

The TBA equations in a torus geometry with no boundary were studied in [1]. In the cylindrical geometry considered here with boundary interactions, the two possible ways of viewing the partition function which correspond to the two possible channels for time evolution are rather asymmetrical, thus one expects the TBA equations for each channel to take different forms.

In the analysis of the previous section, we examined the large L behaviour of the partition function using the boundary states. This corresponds to viewing the partition function in the ‘ R -channel’, where time evolution is in the direction of R . As we now describe, one can obtain the TBA equations in this channel in the limit of large L .

In the situation with non-trivial S-matrix, one may attempt to evaluate (2.2) using the form of the boundary states (2.4) and the Faddeev-Zamolodchikov algebra:

$$\begin{aligned} A^\dagger(\theta)A^\dagger(\theta') &= S(\theta - \theta')A^\dagger(\theta')A^\dagger(\theta), \\ A(\theta)A(\theta') &= S(\theta - \theta')A(\theta')A(\theta), \\ A(\theta)A^\dagger(\theta') &= S(\theta - \theta')A^\dagger(\theta')A(\theta) + \delta(\theta - \theta'). \end{aligned} \tag{3.1}$$

As in the Ising case discussed in the previous section, one finds that the result is ill-defined due to the fact that the formula (2.4) for boundary states is meant in the limit $L \rightarrow \infty$, and this leads to squares of δ -functions. In the Ising case, the regularization (2.13) is adequate since the particles are free and the quantization condition on the momenta does not involve the S-matrix. More generally, one can view the following TBA derivation as a way of regularizing the inner product (2.2) for very large but finite L .

For simplicity, consider a model with a single particle excitation. Our starting point is eq. (2.2) which can be written as

$$Z_{ab} = \sum_{\alpha} \frac{\langle B_a | \alpha \rangle \langle \alpha | B_b \rangle}{\langle \alpha | \alpha \rangle} e^{-RE_{\alpha}}, \tag{3.2}$$

where $|B_a\rangle$ and $|B_b\rangle$ are boundary states and the sum over α stands for a formal sum over all the states of the theory. Due to the form of the boundary states, the states $|\alpha\rangle$ that contribute to the sum are of the form

$$\begin{aligned} |\alpha_{2N}\rangle &= |\theta_N, -\theta_N, \dots, \theta_1, -\theta_1\rangle \\ &= A^\dagger(\theta_N)A^\dagger(-\theta_N) \dots A^\dagger(\theta_1)A^\dagger(-\theta_1)|0\rangle, \end{aligned} \tag{3.3}$$

where $\theta_N > \theta_{N-1} > \dots > \theta_1 > 0$ (inequalities are strict, since due to the condition $S(0) = -1$, the Bethe wave function for particles on a circle of radius L vanishes when two rapidities are equal). Using the Faddeev-Zamolodchikov algebra, one finds that

$$\langle B_a | 2N \rangle = (\delta(0))^N \prod_{i=1}^N \bar{K}_a(\theta_i), \quad (3.4)$$

where the formal expression $(\delta(0))^N$ results from terms $\prod_i \delta(\theta_i - \theta_i)$. It is simple to show (3.4) by introducing pair creation operators $B(\theta) = A(-\theta)A(\theta)$, $B^\dagger(\theta) = A^\dagger(\theta)A^\dagger(-\theta)$, satisfying the algebra

$$\begin{aligned} B(\theta)B(\theta') &= B(\theta')B(\theta), & B^\dagger(\theta)B^\dagger(\theta') &= B^\dagger(\theta')B^\dagger(\theta), \\ B(\theta)B^\dagger(\theta') &= B^\dagger(\theta')B(\theta) + \delta^2(\theta - \theta') \\ &\quad + \delta(\theta - \theta')S(\theta' - \theta)A^\dagger(-\theta')A(-\theta) + \delta(\theta - \theta')S(\theta - \theta')A^\dagger(\theta')A(\theta). \end{aligned} \quad (3.5)$$

One also finds $\langle 2N | 2N \rangle = (\delta(0))^{2N}$, thus,

$$\frac{\langle B_a | 2N \rangle \langle 2N | B_b \rangle}{\langle 2N | 2N \rangle} = \prod_{i=1}^N \bar{K}_a(\theta_i) K_b(\theta_i). \quad (3.6)$$

Let us introduce a density per unit length of pairs of particles $P(\theta)$, so that in (3.2), $E_\alpha = \sum_{i=1}^N 2m \cosh \theta_i$ is replaced with $2mL \int d\theta \cosh(\theta) P(\theta)$. Due to the factorization (3.6), the partition function reads

$$Z \approx \int [dP] \exp \left\{ L \int_0^\infty [\log(\bar{K}_a(\theta) K_b(\theta)) - 2Rm \cosh \theta] P(\theta) d\theta + S([P]) \right\}, \quad (3.7)$$

where $S[P]$ is the entropy of the particle configuration described by the distribution $P(\theta)$. From this equation, we see that the contribution of the boundaries has a natural interpretation as (rapidity dependent) chemical potentials.

The density $P(\theta)$ is constrained by the quantization condition. For the states $|2N\rangle$, this reads

$$e^{imL \sinh(\theta_i)} S(2\theta_i) \prod_{j \neq i} S(\theta_i - \theta_j) S(\theta_i + \theta_j) = 1. \quad (3.8)$$

Introducing densities of particles and holes as usual such that $L[P(\theta) + P^h(\theta)]d\theta$ is the number of allowed rapidities between θ and $\theta + d\theta$, one has the quantization conditions

$$2\pi (P(\theta) + P^h(\theta)) = m \cosh \theta - 2\pi \int_0^\infty [\Phi(\theta - \theta') + \Phi(\theta + \theta')] P(\theta') d\theta', \quad (3.9)$$

where Φ is defined by

$$\Phi(\theta) = -\frac{1}{2i\pi} \frac{d}{d\theta} \ln S(\theta). \quad (3.10)$$

In terms of P, P^h , the entropy takes the usual form:

$$S([P]) = L \int_0^\infty d\theta [(P + P^h) \ln(P + P^h) - P \ln P - P^h \ln P^h]. \quad (3.11)$$

Finally, the TBA equations arise as a saddle point evaluation of (3.7), i.e. the minimization of the free energy, subject to the constraint (3.9). Using standard manipulations, one obtains the leading behaviour of $\ln Z$ for large L :

$$\log Z_{ab} \approx -LE_{ab}^0(R) = \frac{mL}{2\pi} \int_0^\infty \cosh \theta \ln [1 + e^{-\epsilon(\theta)}] d\theta. \quad (3.12)$$

where ϵ satisfies

$$\epsilon(\theta) = 2Rm \cosh \theta - \log(\overline{K}_a K_b) + \Phi_s * \log(1 + e^{-\epsilon}). \quad (3.13)$$

In the previous equation, $\Phi_s \equiv \Phi(\theta - \theta') + \Phi(\theta + \theta')$ and $*$ denotes the convolution. The combination $\overline{K}_a(\theta)K_b(\theta)$ is an even function of θ , since $\overline{K}(\theta) = K(-\theta)$. Thus, we can extend the domain of definition of $\epsilon(\theta)$ to the whole real axis by $\epsilon(-\theta) = \epsilon(\theta)$ and let the integrals run from $-\infty$ to $+\infty$. The equations for the ground state energy may be recast into the following:

$$\begin{aligned} E_{ab}^0(R) &= -\frac{m}{4\pi} \int_{-\infty}^\infty \cosh \theta \log [1 + \lambda_{ab}(\theta) e^{-\epsilon(\theta)}] d\theta, \\ \epsilon(\theta) &= 2Rm \cosh \theta + \int_{-\infty}^\infty \Phi(\theta - \theta') \log [1 + \lambda_{ab}(\theta') e^{-\epsilon(\theta')}] d\theta', \end{aligned} \quad (3.14)$$

where

$$\lambda_{ab}(\theta) = \overline{K}_a(\theta)K_b(\theta).$$

One may now see explicitly how the TBA regularized the inner product (2.2). A multiparticle expansion of the TBA equations (3.12)(3.13) can be obtained by solving them iteratively as follows:

$$\epsilon_{n+1} = \epsilon_0 + \Phi_s * \log(1 + e^{-\epsilon_n}), \quad (3.15)$$

starting from $\epsilon_0 = 2mR \cosh \theta - \log(\overline{K}_a K_b)$. One obtains

$$Z_{ab} = 1 + \frac{mL}{2\pi} (D_1 + D_2 + \dots) + \frac{1}{2} \left(\frac{mL}{2\pi} \right)^2 D_1^2 + \dots, \quad (3.16)$$

where

$$\begin{aligned}
D_1 &= \int_{-\infty}^{\infty} d\theta \cosh \theta e^{-2mR \cosh \theta} \overline{K}_a(\theta) K_b(\theta), \\
D_2 &= -\frac{1}{2} \int_{-\infty}^{\infty} d\theta \cosh \theta e^{-4mR \cosh \theta} \overline{K}_a^2(\theta) K_b^2(\theta) + \frac{1}{2\pi} \int_{-\infty}^{\infty} d\theta d\theta' e^{-2mR(\cosh \theta + \cosh \theta')} \\
&\quad (\Phi(\theta + \theta') + \Phi(\theta - \theta')) \overline{K}_a(\theta') K_b(\theta') \overline{K}_a(\theta) K_b(\theta).
\end{aligned} \tag{3.17}$$

On the other hand, evaluating (2.2) using the Faddeev-Zamolodchikov algebra, one obtains

$$\begin{aligned}
Z_{ab} &= 1 + \int d\theta \delta(0) \{ \overline{K}_a K_b e^{-2mR \cosh \theta} - \frac{1}{2} \overline{K}_a^2 K_b^2 e^{-4mR \cosh \theta} \} + \\
&\quad + \frac{1}{2} \int d\theta d\theta' \delta^2(0) \overline{K}_a(\theta) K_b(\theta) \overline{K}_a(\theta') K_b(\theta') e^{-2mR(\cosh \theta + \cosh \theta')} + \dots
\end{aligned} \tag{3.18}$$

Comparing (3.16) with (3.18), one sees that the single δ -functions in (3.18) were regulated as in (2.13), however the $\delta^2(0)$ is regulated in a more complicated fashion involving the S-matrix:

$$\delta(\theta - \theta) \delta(\theta' - \theta') \rightarrow \left(\frac{mL}{2\pi} \right)^2 \cosh \theta \cosh \theta' + \frac{mL}{2\pi^2} \cosh \theta (\Phi(\theta + \theta') + \Phi(\theta - \theta')). \tag{3.19}$$

As an explicit example of our discussion, we will consider the $\varphi_{1,3}$ massive deformation of the minimal non-unitarity conformal model $\mathcal{M}_{3,5}$, in the R-channel. The central charge is $c = -3/5$ and there are four primary fields of chiral dimensions $(0, -\frac{1}{20}, \frac{1}{5}, \frac{3}{4})$. The massive phase of this deformation has only one massive particle with the two-particle elastic S-matrix given by $S(\theta) = -i \tanh \frac{1}{2} (\theta - i\frac{\pi}{2})$ [6]. For the boundary reflection amplitudes, solutions of the equations

$$\begin{aligned}
\mathcal{R}(\theta) \mathcal{R}(-\theta) &= 1, \\
\mathcal{R}\left(\frac{i\pi}{2} - \theta\right) &= S(2\theta) \mathcal{R}\left(\frac{i\pi}{2} + \theta\right),
\end{aligned} \tag{3.20}$$

we have two minimal solutions, given by

$$\begin{aligned}
\mathcal{R}_1(\theta) &= -i \tanh \left(\frac{\theta}{2} - \frac{i\pi}{4} \right) \frac{\sinh \left(\frac{\theta}{2} - \frac{i\pi}{8} \right)}{\sinh \left(\frac{\theta}{2} + \frac{i\pi}{8} \right)}, \\
\mathcal{R}_2(\theta) &= i \coth \left(\frac{\theta}{2} - \frac{i\pi}{4} \right) \frac{\sinh \left(\frac{\theta}{2} - \frac{i\pi}{8} \right)}{\sinh \left(\frac{\theta}{2} + \frac{i\pi}{8} \right)},
\end{aligned} \tag{3.21}$$

and from those, we can compute $K(\theta) = \mathcal{R}\left(\frac{i\pi}{2} - \theta\right)$. We now want to extract the ultraviolet limit of the ground state energy for different boundary conditions associated to these

amplitudes. The kernel entering the TBA equation is given by $\Phi(\theta) = 1/\cosh \theta$. As in analogous TBA computation, the ultraviolet limit of the ground state energy is ruled by the "kink" solution of the pseudoenergy $\epsilon(\theta)$, i.e. the universal function which flattens in the central region $-\log(2/r) \ll \theta \ll \log(2/r)$ to the constant value ϵ_{ab}^0 , solution in this case of the algebraic equation

$$\epsilon_{ab}^0 = -\frac{1}{2} \log \left[1 + \lambda_{ab}(\infty) e^{-\epsilon_{ab}^0} \right]. \quad (3.22)$$

In fact, in the ultraviolet limit $mR \rightarrow 0$, we have

$$E_{ab}^0 \approx -\frac{1}{4\pi R} \int_{\epsilon_{ab}^0}^{\infty} d\epsilon \left\{ \frac{\epsilon \lambda_{ab}(\infty) e^{-\epsilon}}{1 + \lambda_{ab}(\infty) e^{-\epsilon}} + \log [1 + \lambda_{ab}(\infty) e^{-\epsilon}] \right\}. \quad (3.23)$$

Choosing boundary conditions of the type (1,1) or (2,2) on both sides of the strip, we have $\lambda_{11}(\infty) = \lambda_{22}(\infty) = 1$ and for the constant value of the pseudoenergy we have

$$\epsilon_{11}^0 = \epsilon_{22}^0 = -\log \left(\frac{1 + \sqrt{5}}{2} \right), \quad (3.24)$$

and as a final result

$$E_{11}^0 = E_{22}^0 \approx -\frac{\pi}{40R}. \quad (3.25)$$

On the other hand, choosing boundary conditions of the type (1,2), we have $\lambda_{12}(\infty) = -1$.

In this situation we have

$$\epsilon_{12}^0 = -\log \left(\frac{-1 + \sqrt{5}}{2} \right), \quad (3.26)$$

and correspondingly

$$E_{12}^0(R) \approx \frac{\pi}{40R}. \quad (3.27)$$

In terms of CFT, in the first two cases the boundary conditions select as dominant contribution in the partition function the one of the conformal chiral field σ of anomalous dimension $\Delta = -\frac{1}{20}$ whereas in the last case, the leading contribution comes from the identity family.

In closing this section, observe that by taking the limit $R \rightarrow 0$ in the above equations (3.12)(3.13), we obtain the leading behaviour for the scalar product of boundary states

$$\begin{aligned} \ln \langle B_a | B_b \rangle &\approx \frac{mL}{4\pi} \int_{-\infty}^{\infty} \cosh \theta \ln \left[1 + \lambda_{ab} e^{-\epsilon(\theta)} \right] d\theta, \\ \epsilon(\theta) &= \int_{-\infty}^{\infty} \Phi(\theta - \theta') \ln \left[1 + \lambda_{ab}(\theta) e^{-\epsilon(\theta)} \right] d\theta'. \end{aligned} \quad (3.28)$$

4. Normalizations and TBA in the L-channel

We can also discuss in more detail the overall normalization alluded to in (2.4). There, $|0\rangle$ is the ground state of the massive theory, which in the renormalized theory is simply a state without any excitation. Although the correspondence between particle states of the massive theory and conformal states in the UV limit is not totally known, the ground states usually match, that is in the UV limit, $|0\rangle$ in (2.4) goes to the conformal ground state $|0\rangle_c$, while all the states with particles go to combination of fields and descendants with higher conformal weights. Therefore, if the boundary state (2.4) corresponds to some state $|B\rangle_c$ in the conformal theory, we expect

$$g = \langle 0|B(m)\rangle = \langle 0|B\rangle_c \text{ for any } m. \quad (4.1)$$

Let us further investigate this identity. In the “L-channel”, time evolution is along the circumference of the cylinder of length L , and the Hilbert space is constructed along the finite segment of length R . In the limit of large R , TBA equations for the partition function were obtained in [7]. In this channel, the particles are confined to the segment of length R and reflect off the boundary with reflection amplitude $\mathcal{R}_a(\theta)$ at one boundary, and $\mathcal{R}_b(\theta)$ at the other boundary. The quantization condition on the momenta involves in this case the boundary reflection amplitudes:

$$e^{2imR \sinh(\theta_i)} \left[\prod_{j \neq i} S(\theta_i - \theta_j) S(\theta_i + \theta_j) \right] \mathcal{R}_a(\theta_i) \mathcal{R}_b(\theta_i) = 1, \quad (4.2)$$

where $\theta_i > 0$. Introduce the density of particles and holes as usual, so $2R[P(\theta) + P^h(\theta)]d\theta$ is the number of allowed rapidities between θ and $\theta + d\theta$. It is also convenient to let rapidities run from $-\infty$ to ∞ , defining the densities for negative rapidities by parity, so (4.2) reads

$$2\pi(P(\theta) + P^h(\theta)) = m \cosh \theta - 2\pi \int_{-\infty}^{\infty} \Phi(\theta - \theta') P(\theta') + \frac{\Theta_{ab}(\theta)}{2R}, \quad (4.3)$$

where Φ is as before, and

$$\Theta_{ab}(\theta) = \frac{1}{i} \frac{d}{d\theta} \log(\mathcal{R}_a \mathcal{R}_b) - \frac{1}{i} \frac{d}{d\theta} \log S(2\theta) - 2\pi \delta(\theta), \quad (4.4)$$

and we used the fact that, by unitarity, Θ_{ab} is an even function. In the latter expression, the δ term arises because vanishing rapidities, although possible solutions of (4.2), are not acceptable since the corresponding wave function vanishes exactly.

From here the standard TBA procedure leads to the following result:

$$\begin{aligned}\log Z_{ab} &\approx \frac{1}{4\pi} \int_{-\infty}^{\infty} [2mR \cosh \theta + \Theta_{ab}(\theta)] \log [1 + e^{-\epsilon(\theta)}] d\theta, \\ \epsilon(\theta) &= mL \cosh \theta + \int_{-\infty}^{\infty} \Phi(\theta - \theta') \log [1 + e^{-\epsilon(\theta')}] d\theta'.\end{aligned}\tag{4.5}$$

The Θ term in (4.5) is R independent and may be considered as a boundary free energy term. As discussed in [7] the TBA actually reproduces the boundary free energy up to an additive constant. This is because, when considering contributions of order one, one should take care of various corrections like corrections to the Stirling formula in evaluating the entropy, or loop corrections to the saddle point, which we have discarded here. Also the correspondence between the entropy of the field theory and the one computed using particles might involve some constant, for instance if the particles describe kinks and an overall choice of configuration is left out.

Now we can interpret this boundary free energy easily. Indeed in the large R limit, the next to leading behaviour of the partition function (2.2) is

$$Z_{ab} \approx \langle B_a | 0 \rangle \langle 0 | B_b \rangle e^{-RE_0(R)},\tag{4.6}$$

up to exponentially small terms, where E_0 is the ground state energy of periodic Hamiltonian. Therefore we have

$$\ln \langle B_a | 0 \rangle \langle 0 | B_b \rangle = \frac{1}{4\pi} \int_{-\infty}^{\infty} \Theta_{ab}(\theta) \ln [1 + e^{-\epsilon(\theta)}] d\theta + \text{cst}.\tag{4.7}$$

In the IR limit given by $L \rightarrow \infty$, ϵ goes to infinity and the first term on the rhs of (4.7) vanishes. In the UV limit, expressed by $L \rightarrow 0$, on the contrary ϵ goes to a constant. The statement (4.1) translates therefore into the condition

$$\int_{-\infty}^{\infty} \Theta(\theta) d\theta = 0.\tag{4.8}$$

This condition is indeed true, as can be easily seen by using (4.4) and the cross unitarity relation

$$\mathcal{R}\left(i\frac{\pi}{2} - \theta\right) = S(2\theta) \mathcal{R}\left(i\frac{\pi}{2} + \theta\right),\tag{4.9}$$

together with analyticity.

As we will see in the next section, for finite L , g can be a non-trivial function of L .

5. Massless Theory in the Bulk

An interesting situation is when the bulk theory is massless, but the conformal invariance is broken by the boundary interactions. The previous formalism carries over easily to that case. This time the bulk theory is described by massless particles which are left or right moving with trivial left-right scattering. Parametrize the right (left) moving particles by rapidities so that $E = p = \frac{m}{2}e^\theta$ (resp. $E = -p = \frac{m}{2}e^{-\theta}$) (observe that opposite momenta still correspond to opposite rapidities). Then the left-left and right-right scattering are described by an identical S matrix in terms of rapidities, and one can introduce as before

$$\Phi(\theta) = -\frac{1}{2i\pi} \frac{d}{d\theta} \ln S_{LL}(\theta). \quad (5.1)$$

Now all rapidities in $[-\infty, \infty]$ are allowed, so the quantization condition reads

$$2\pi(P(\theta) + P^h(\theta)) = \frac{m}{2}e^\theta - 2\pi \int_{-\infty}^{\infty} \Phi(\theta - \theta') P(\theta') d\theta', \quad (5.2)$$

and for the entropy (3.11) the integration runs from $-\infty$ to ∞ . The energy reads $E = \sum_{i=1}^N m e^{\theta_i}$. To have a breaking of the conformal invariance and a corresponding flow, one needs to put an energy scale at the boundary, so the boundary S matrix depends on some extra parameter (e.g. a magnetic field), corresponding to an amplitude $K_a(\theta - \theta_{B_a})$. The boundary state is then a combination of massless states involving pairs of left and right moving particles with opposite momenta (hence opposite rapidities using the above parametrization)

$$|B_a\rangle = g \exp \left[\int_{-\infty}^{\infty} \frac{d\theta}{2\pi} K_a(\theta - \theta_{B_a}) A_L^\dagger(-\theta) A_R^\dagger(\theta) \right] |0\rangle. \quad (5.3)$$

Then (3.6) still holds. In the “R-channel”, the maximization of the free energy leads now to

$$\begin{aligned} E_{ab}^0 &= -\frac{m}{4\pi} \int_{-\infty}^{\infty} e^\theta \ln \left[1 + \lambda_{ab}(\theta) e^{-\epsilon(\theta)} \right] d\theta \\ \epsilon(\theta) &= m R e^\theta + \int_{-\infty}^{\infty} \Phi(\theta - \theta') \ln \left[1 + \lambda_{ab}(\theta') e^{-\epsilon(\theta')} \right] d\theta', \end{aligned} \quad (5.4)$$

where

$$\lambda_{ab}(\theta) = \overline{K_a}(\theta - \theta_{B_a}) K_b(\theta - \theta_{B_b}). \quad (5.5)$$

It is also possible to compute the boundary entropies by using the TBA in the “L-channel”, as above. One finds

$$\ln \langle B_a | 0 \rangle \langle 0 | B_b \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Theta_{ab}(\theta) \ln \left[1 + e^{-\epsilon(\theta)} \right] d\theta + \text{cst}, \quad (5.6)$$

where

$$\epsilon(\theta) = mL \frac{e^\theta}{2} + \int_{-\infty}^{\infty} \Phi(\theta - \theta') \ln \left[1 + e^{-\epsilon(\theta')} \right] d\theta', \quad (5.7)$$

and

$$\Theta_{ab}(\theta) = \frac{1}{i} \frac{d}{d\theta} \ln [\mathcal{R}_a(\theta - \theta_{B_a}) \mathcal{R}_b(\theta - \theta_{B_b})]. \quad (5.8)$$

In the last expression compared to (4.4) the term $S(2\theta)$ has disappeared because it would arise from LR scattering which is rapidity independent. Similarly, vanishing momentum corresponds now to vanishing energy and the corresponding term can be discarded.

Despite the close similarity of the massless and massive TBA equations, what is meant by ultraviolet and infrared limit in the massless case is different from the massive situation. In fact, instead of the bulk mass parameter, we now vary the boundary energy scale me^{θ_B} (m just fixes the overall scale and can be considered constant) which corresponds physically to varying the boundary interaction. Boundary states in general will correspond to conformal states both in the UV and in the IR. These conformal states in general will have different scalar products with the conformal ground state, and the boundary entropy will vary along the flow. Accordingly, eq. (4.8) does not hold. Physically, the boundary state (5.3) can now have varying scalar product with the ground state because conformal states can be represented as combinations of massless particle states.

To illustrate this situation, consider the Ising model at $T = T_c$ but with a boundary magnetic field. This induces a flow from the free to the fixed boundary conditions. The flow is described by a massless boundary matrix

$$\mathcal{R}(\theta - \theta_B) = -i \tanh \left(\frac{\theta - \theta_B}{2} - \frac{i\pi}{4} \right). \quad (5.9)$$

Using (5.6) and setting $g = \langle 0|B \rangle$, one finds

$$\ln g = \int_{-\infty}^{\infty} \frac{\ln [1 + e^{-e^\theta}]}{\cosh(\theta - \ln L/L_B)} \frac{d\theta}{2\pi} + \text{cst}, \quad (5.10)$$

where $L_B = \frac{2}{m} e^{-\theta_B}$. This is easily evaluated

$$g = \text{cst} \frac{\sqrt{2\pi}}{\Gamma(a + 1/2)} \left(\frac{a}{e} \right)^a, \quad (5.11)$$

where $a = L/L^B$. The UV limit is $L \rightarrow 0$ and $g \rightarrow \text{cst} \sqrt{2}$, while the IR limit is $L \rightarrow \infty$ and $g \rightarrow \text{cst} 1$. Therefore we find $g^{UV}/g^{IR} = \sqrt{2} = g^{free}/g^{fixed}$ in agreement with [8].

The above characterization of the boundary state involving massless particles is appropriate for very large L . For finite L another characterization is in principle possible which does not involve massless particles, as we now describe. Viewing the partition function in the R-channel, the Hamiltonian and the Hilbert space are the same as for the conformal field theory on a cylinder. Therefore, the boundary states can be expressed in terms of conformal operators even though the boundary states themselves break the conformal invariance. Furthermore, since the conformal field theory is well defined for finite L , one can obtain exact partition functions for both finite L and R . For the Ising case, this was studied by Chatterjee [9]¹. For perturbed minimal models, these boundary states were characterized in [10]. In this section, we show how formulas of the previous sections in part characterize the boundary states in the simple case of the Ising model.

Let $\psi(z), \bar{\psi}(\bar{z})$ denote the left and right components of the Ising free fermion field, where $z = t + ix$, $\bar{z} = t - ix$. On a cylinder with x varying along the circumference of length L and t varying along its length R , the mode expansions read

$$\begin{aligned}\psi(z) &= \sum_r \psi_r \exp(-2\pi r z/L) \\ \bar{\psi}(\bar{z}) &= \sum_r \bar{\psi}_r \exp(-2\pi r \bar{z}/L),\end{aligned}\tag{5.12}$$

where $r \in \mathbb{Z} + 1/2$ ($r \in \mathbb{Z}$) in the Neveu-Schwarz (Ramond) sector. The interpolating boundary condition takes the form[3]:

$$\frac{d}{dx} (\psi - \bar{\psi}) - i \frac{h^2}{2} (\psi + \bar{\psi}) = 0 \quad (t = 0).\tag{5.13}$$

Inserting the mode expansions (5.12) into (5.13), one obtains

$$\bar{\psi}_n = a_n \psi_{-n},\tag{5.14}$$

where

$$a_n = \frac{n - a}{n + a}, \quad a = \frac{h^2 L}{4\pi}.\tag{5.15}$$

The equation (5.14) implies that the boundary state must satisfy

$$(\bar{\psi}_n - a_n \psi_{-n}) |B\rangle = 0.\tag{5.16}$$

¹ We obtained the results in [9] independently, in a manner presented below.

This fixes the boundary states up to a constants g , which can depend on L and h :

$$\begin{aligned} |B >^{NS} &= g_+ \exp \left(\sum_{r \in \mathbb{Z} + 1/2, r > 0} a_r \bar{\psi}_{-r} \psi_{-r} \right) |0 > \\ |B >^R &= g_- \exp \left(\sum_{n \in \mathbb{Z}, n > 0} a_n \bar{\psi}_{-n} \psi_{-n} \right) |\sigma >, \end{aligned} \quad (5.17)$$

where $|\sigma >$ is the spin field state of dimension $1/8$ in the Ramond sector.

The factors g_{\pm} depend only on L and h , and are characterized by $g_+ = \langle 0 | B >^{NS}$, $g_- = \langle 0 | B >^R$. Setting the length R to zero and one of the boundary scattering matrices \mathcal{R}_a to 1 in (4.5), one sees that the g -factors correspond to the boundary free energy term (Θ term). Since the S-matrix is -1 , the integral equation just implies $\epsilon = mL \cosh \theta$. Thus g_{\pm} are just the massless limit of the expressions:

$$\log g_{\pm} = \lim_{m \rightarrow 0} \int_0^{\infty} \frac{d\theta}{2\pi} \left(-i \frac{d}{d\theta} \log R(\theta) - \pi \delta(\theta) \right) \log(1 \pm e^{-mL \cosh \theta}). \quad (5.18)$$

The extra minus sign in g_- versus g_+ is easily obtain by including a $(-1)^F$ in the TBA computation, where F is fermion number.

One has $\mathcal{R}(\theta) = K(\frac{i\pi}{2} - \theta)$, where K is given in (2.7). Defining $x = mL \cosh \theta / 2\pi a$ and letting $m \rightarrow 0$, one obtains

$$\begin{aligned} \log g_+ &= \frac{1}{\pi} \int_0^{\infty} \frac{1}{1+x^2} \log(1 + e^{-2\pi a x}) \\ \log g_- &= \frac{1}{\pi} \int_0^{\infty} \frac{1}{1+x^2} \log(1 - e^{-2\pi a x}) + \frac{1}{4} \log 2. \end{aligned} \quad (5.19)$$

The extra term in $\log g_-$ arises from the contribution

$$\lim_{m \rightarrow 0} \int_0^{\infty} \frac{d\theta}{2\pi} \left(\frac{1}{\cosh \theta} - \pi \delta(\theta) \right) \log(1 - e^{-mr \cosh \theta}) = \frac{1}{4} \log 2. \quad (5.20)$$

(The $m \rightarrow 0$ limit is obtained from $\log(1 - e^{-mr \cosh \theta}) \approx \log mr + \log \cosh \theta$; the $\log mr$ term vanishes, and the $\log \cosh \theta$ term gives (5.20)). The above integrals are easily done²:

$$g_+ = \frac{\sqrt{2\pi}}{\Gamma(a + 1/2)} \left(\frac{a}{e} \right)^a, \quad g_- = 2^{1/4} \frac{\sqrt{2\pi a}}{\Gamma(a + 1)} \left(\frac{a}{e} \right)^a. \quad (5.21)$$

² The results for these integrals presented in [11] are off by a minus sign.

The complete boundary state is a linear combination of the NS and R boundary states:

$$|B_h\rangle = \frac{1}{\sqrt{2}} (|B\rangle^{NS} + \text{sign}(h)|B\rangle^R). \quad (5.22)$$

The partition function with magnetic fields h, h' on the boundaries is now easily computed as

$$Z_{h',h} = \langle B_{h'} | e^{-HR} | B_h \rangle, \quad (5.23)$$

where H is the hamiltonian on the cylinder:

$$H = \frac{2\pi}{L} \left(L_0 + \bar{L}_0 - \frac{c}{12} \right). \quad (5.24)$$

Using the formula (2.20), one obtains:

$$\begin{aligned} Z_{h',h} = & \frac{1}{2} \tilde{q}^{-1/48} g_+(h) g_+(h') \prod_{n=0}^{\infty} \left(1 + a_{n+1/2}(h') a_{n+1/2}(h) \tilde{q}^{n+1/2} \right) \\ & + \frac{1}{2} \text{sign}(hh') g_-(h) g_-(h') \tilde{q}^{1/24} \prod_{n=0}^{\infty} (1 + a_n(h') a_n(h) \tilde{q}^n), \end{aligned} \quad (5.25)$$

where $\tilde{q} = \exp(-4\pi R/L)$.

The free and fixed boundary conditions at $h = 0$ and $h = \pm\infty$ respectively are conformal. In these limits, the above partition function corresponds precisely to the modular transformation of the formula (2.21). To show this, recall that in the L-channel, the partition function is [5]:

$$Z_{ab} = \sum_{i,j} n_{ab}^i S_i^j \chi_j(\tilde{q}), \quad (5.26)$$

where S determines how the characters transform, and $\tilde{q} = \exp(-4\pi R/L)$. As $h \rightarrow 0$ ($h \rightarrow \pm\infty$), $a_n = 1$ ($a_n = -1$). Also,

$$\begin{aligned} h \rightarrow 0 : & \quad g_+ = \sqrt{2}, \quad g_- = 0 \\ h \rightarrow \pm\infty : & \quad g_+ = 1, \quad g_- = 2^{+1/4}. \end{aligned} \quad (5.27)$$

The Virasoro characters have the infinite product expressions [12]:

$$\begin{aligned} \chi_0(q) + \chi_{1/2}(q) &= q^{-1/48} \prod_{n=0}^{\infty} (1 + q^{n+1/2}) \\ \chi_0(q) - \chi_{1/2}(q) &= q^{-1/48} \prod_{n=0}^{\infty} (1 - q^{n+1/2}) \\ \chi_{1/16}(q) &= q^{1/24} \prod_{n=0}^{\infty} (1 + q^n). \end{aligned} \quad (5.28)$$

Let f denote free, and \pm denote the fixed boundary conditions as $h \rightarrow \pm\infty$. Then in these conformal limits, from (5.25) one recovers (5.26), where the only non-zero n_{ab}^i are $n_{\pm\pm}^0, n_{ff}^0, n_{ff}^{1/2}, n_{\pm\mp}^{1/2}, n_{\pm f}^{1/16}$, and S is given by

$$S = \frac{1}{2} \begin{pmatrix} 1 & 1 & \sqrt{2} \\ 1 & 1 & -\sqrt{2} \\ \sqrt{2} & -\sqrt{2} & 0 \end{pmatrix}. \quad (5.29)$$

(The rows and columns refer to the characters in the order $0, 1/2, 1/16$.) For instance,

$$Z_{+-} = Z_{h'=\infty, h=-\infty} = \frac{1}{2} \left(\chi_0(\tilde{q}) + \chi_{1/2}(\tilde{q}) - \sqrt{2}\chi_{1/16}(\tilde{q}) \right). \quad (5.30)$$

Part II: The sine-Gordon case

6. TBA in the L channel.

6.1. The inhomogeneous 6-vertex model with boundary fields

Notations closely follow those of [7]. One starts from the inhomogeneous 6-vertex model with boundary fields h as a regularization of the boundary sine-Gordon model with Dirichlet boundary conditions.

In the inhomogeneous antiferromagnetic 6-vertex model with anisotropy parameter γ , one gives an alternating imaginary part $\pm i\Lambda$ to the spectral parameter on alternating vertices [13,14]. The scaling limit is given by taking $\Lambda \rightarrow \infty$, $N \rightarrow \infty$, and the lattice spacing $\Delta \rightarrow 0$, such that $R \equiv N\Delta$ remains finite. In the bulk, this provides a regularization of the sine-Gordon model with Lagrangian

$$L_{\text{SG}} = \int_0^R dx \left[\frac{1}{2}(\partial\phi)^2 + \mu^2 \cos \beta_{\text{SG}}\phi \right] \quad (6.1)$$

where $\mu \propto \frac{1}{\Delta} \exp(-\text{const}\Lambda)$, $\beta_{\text{SG}}^2 = 8(\pi - \gamma)$, and the field is fixed at $x = 0$ and $x = R$ (Dirichlet boundary conditions) to a value that is simply related to h .

The wave function of the inhomogeneous six-vertex model can be expressed in terms of a set of “roots” α_j , where $j = 1 \dots n$. They must be solutions of the set of equations [15,16]

$$N[f(\alpha_j + \Lambda, \gamma) + f(\alpha_j - \Lambda, \gamma)] + 2f(\alpha_j, \gamma H) = 2\pi l_j + \sum_{m=1, m \neq j}^n [f(\alpha_j - \alpha_m, 2\gamma) + f(\alpha_j + \alpha_m, 2\gamma)], \quad (6.2)$$

where l_j is an integer and all α_j are positive. The function f is defined as

$$f(a, b) = 2 \tan^{-1} \left(\cot \frac{b}{2} \tanh a \right)$$

and

$$H \equiv \frac{1}{\gamma} f(i\gamma, -i \ln(h + \cos \gamma)). \quad (6.3)$$

For $h = 0$, $\gamma H = \pi - \gamma$. By construction of the Bethe-ansatz wave function, $\alpha_j > 0$. Even though there is a solution of (6.2) with one vanishing root for any N and n , we emphasize that $\alpha_j = 0$ is **not** allowed because the wave function vanishes identically in this case. Observe that equations (6.2) are formally satisfied as well by the opposite of the roots,

$-\alpha_j$. Often in what follows we shall consider that the roots take both signs in order to rewrite equations in a way which is similar to the bulk case.

The solutions of these equations are quite intricate for arbitrary γ [17]. For simplicity, we restrict to the case $\gamma = \frac{\pi}{t}$ where t is an integer, and restrict to the choice $\epsilon = -1$. In the sine-Gordon model, this falls in the repulsive regime. We make the standard assumption that all the solutions of interest are collections of “ k -strings” for $k = 1, 2 \dots t-1$ and antistrings a [17]. A k -string is a group of α_j in the pattern $\alpha^{(k)} - i\pi(k-1), \alpha^{(k)} - i\pi(k-3), \dots, \alpha^{(k)} + i\pi(k-1)$ where $\alpha^{(k)}$ is real. The antistring has $\alpha_j = \alpha^{(a)} + i\pi$, where $\alpha^{(a)}$ is real.

The thermodynamic limit is obtained by sending $N \rightarrow \infty$. In this case, we can define densities of the different kinds of solutions. The number of allowed solutions of (6.2) of type k in the interval $(\alpha, \alpha + d\alpha)$ is $2N(\rho_k(\alpha) + \rho_k^h(\alpha))d\alpha$, where ρ_k is the density of “filled” solutions (those which appear in the sum in the right-hand-side of (6.2)) and ρ_k^h is the density of “holes” (unfilled solutions). The densities ρ_a and ρ_a^h are defined likewise for the antistring. The “bare” Bethe ansatz equations follow from taking the derivative of (6.2). For $\gamma = \pi/t$ they can be written in the form:

$$\begin{aligned} 2\pi(\rho_k + \rho_k^h) &= a_k(\alpha) - \dot{\phi}_{k,t-1} * \rho_a + \sum_{l=1}^{t-1} \dot{\phi}_{kl} * \rho_l + \frac{1}{2N} u_k \\ 2\pi(\rho_a + \rho_a^h) &= 2\pi(\rho_{t-1} + \rho_{t-1}^h) + \frac{1}{2N}(u_a - u_{t-1}) \end{aligned} \quad (6.4)$$

where $*$ denotes convolution:

$$f * g(\alpha) \equiv \int_{-\infty}^{\infty} d\alpha' f(\alpha - \alpha') g(\alpha').$$

These densities are originally defined for $\alpha > 0$, but the equations allow us to define $\rho_k(-\alpha) \equiv \rho_k(\alpha)$ in order to rewrite the integrals to go from $-\infty$ to ∞ . The kernels in these equations are defined most easily in terms of their Fourier transforms

$$\widehat{f}(x) = \int_{-\infty}^{\infty} \frac{d\alpha}{2\pi} e^{i\alpha x/\pi} f(\alpha), \quad f(\alpha) = \frac{t}{\pi} \int_{-\infty}^{\infty} e^{-i\alpha x/\pi} \widehat{f}(x) dx. \quad (6.5)$$

One has

$$\widehat{\phi}_{kl}(x) = \delta_{ab} - 2 \frac{\cosh x \sinh(t-k)x \sinh lx}{\sinh x \sinh tx}, \quad (6.6)$$

for $k \geq l$ with $\dot{\phi}_{lk} = \dot{\phi}_{kl}$, and

$$\begin{aligned}\hat{a}_k &= \frac{\sinh(t-k)x}{\sinh tx} \cos \Lambda tx / \pi \\ \hat{u}_k &= 2 \frac{\sinh(t-H)x \sinh kx}{\sinh x \sinh tx} + \frac{\sinh(t-2k)x/2}{\sinh tx/2} - 1 \\ \hat{u}_a &= 2 \frac{\sinh Hx}{\sinh tx} - \frac{\sinh(t-2)x/2}{\sinh tx/2} - 1,\end{aligned}\tag{6.7}$$

with in particular $a_1(\alpha) = \frac{1}{2} \left[\dot{f}(\alpha + \Lambda, \gamma) + \dot{f}(\alpha - \Lambda, \gamma) \right]$, $\phi_{11}(\alpha) = -f(\alpha, 2\gamma)$. The boundary manifests itself in the first term in u_k ; notice that even for $h = 0$, it still modifies the equations. A few technicalities account for the other terms (these are relevant here because we are interested in subleading boundary effects). The second term in u_k arises from the fact that the sum in (6.2) does not include the term $m = j$; the integration over densities includes such a contribution and so it must be subtracted off by hand. The third term in u_k arises because ρ and ρ^h are defined for allowed solutions, while as already explained, $\alpha = 0$ is not allowed because it does not give a valid wavefunction. Since it is a valid solution of (6.2) but is not included in the densities, we must subtract an explicit $\frac{2\pi}{2N}\delta(\alpha)$ (corresponding to $1/2N$ in Fourier space). Explicitly, one has

$$u_1 = 2\dot{f}(\alpha, \gamma H) + 2\dot{f}(2\alpha, 2\gamma) - 2\pi\delta(\alpha).\tag{6.8}$$

For compactness we rewrite (6.4) as

$$2\pi\epsilon^{(k)}(\rho_k + \rho_k^h) = a_k(\alpha) + \sum_l \dot{\phi}_{kl} * \rho_l + \frac{1}{2N}u_k,\tag{6.9}$$

where $\epsilon^{(k)} = -1$ for the antistring.

The energy reads, with proper hamiltonian normalization,

$$\frac{E^{latt}}{2N} = -\frac{1}{t} \sum_k \int_{-\infty}^{\infty} a_k(\alpha) \rho_k(\alpha) d\alpha.\tag{6.10}$$

It is easy to write the thermodynamic Bethe ansatz for this model. One finds that the TBA equations, since they are obtained by a variational method, do not depend on boundary terms, and read as usual

$$-\frac{2}{t}a_k(\alpha) = T \ln(1 + e^{\epsilon_k}) - T \sum_l \epsilon^{(l)} \frac{A_{kl}}{2\pi} * \ln(1 + e^{-\epsilon_l}),\tag{6.11}$$

where

$$A_{kl}(\alpha) = 2\pi\epsilon^{(k)}\delta_{kl}\delta(\alpha) - \dot{\phi}_{kl}. \quad (6.12)$$

The free energy does depend on the boundary term and reads

$$\begin{aligned} F^{latt} = & -TN \sum_k \int_{-\infty}^{\infty} \epsilon^{(k)} a_k(\alpha) \ln(1 + e^{-\epsilon_k}) \frac{d\alpha}{2\pi} \\ & - \frac{T}{2} \sum_k \int_{-\infty}^{\infty} \epsilon^{(k)} u_k \ln(1 + e^{-\epsilon_k}) \frac{d\alpha}{2\pi}. \end{aligned} \quad (6.13)$$

In the above formulas the temperature T corresponds in the two-dimensional point of view to having a cylinder of radius $L = 1/T$. We can deduce from this result the ground state energy. Indeed recall that the ground state is obtained by $\rho_k = 0, k \neq 1$ and $\rho_1^h = 0$ so

$$E^{latt} = -N \int_{-\infty}^{\infty} a_1(\alpha) |\epsilon_1^-| \frac{d\alpha}{2\pi} - \frac{1}{2} \int_{-\infty}^{\infty} u_1 |\epsilon_1^-| \frac{d\alpha}{2\pi}, \quad (6.14)$$

where from (6.11) we have

$$\hat{\epsilon}_1^- = -\frac{1}{t} \frac{\cos(\Lambda tx/\pi)}{\cosh x}. \quad (6.15)$$

Replacing and using (6.7) we find

$$\begin{aligned} E_{bulk}^{latt} &= -\frac{N}{\pi} \int_{-\infty}^{\infty} \cos^2\left(\frac{\Lambda tx}{\pi}\right) \frac{\sinh(t-1)x}{\cosh x \sinh tx} \\ E_{bdry}^{latt} &= -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\cos(\Lambda tx/\pi)}{\cosh x} \left(2 \frac{\sinh(t-H)x}{\sinh tx} + \frac{\sinh(t-2)x/2}{\sinh tx/2} - 1 \right), \end{aligned} \quad (6.16)$$

where we used the formula

$$\int_{-\infty}^{\infty} a(\alpha)b(\alpha)d\alpha = 2t \int_{-\infty}^{\infty} \hat{a}(x)\hat{b}(-x)dx.$$

6.2. Boundary energy and entropy of the SG model with Dirichlet boundary conditions

In the continuum limit $\Lambda \rightarrow \infty$ the energy contains various terms. We keep only the finite part which is obtained by closing the above integrals in the upper half plane and selecting the pole at $x = i\frac{\pi}{2}$, leading to

$$\begin{aligned} E_{bulk} &= R \frac{m^2}{4} \cot \frac{t\pi}{2} \\ E_{bdry} &= -\frac{m}{2} \left(2 \frac{\sin(t-H)\pi/2}{\sin t\pi/2} - \cot \frac{t\pi}{4} - 1 \right), \end{aligned} \quad (6.17)$$

where m is the soliton mass,

$$m = 2e^{-t\Lambda/2}. \quad (6.18)$$

All these results trivially generalize to the case of two different boundary fields by splitting the H dependent terms into the sum of an H and an H' term. The bulk result agrees with what is obtained by other methods [18],[19]. For completeness it might be useful to write the boundary energy in terms of the parameter ξ of [3]:

$$E_{bdry} = -\frac{m}{2} \left(\frac{\cos(t-1)\xi}{\sin t\pi/2} + \frac{\cos(t-1)\xi'}{\sin t\pi/2} - \cot t\pi/4 - 1 \right). \quad (6.19)$$

As in the bulk case, when t is even, there are additional logarithmic terms.

We can as well take the continuum limit of the lattice Bethe equations to get the TBA equations for the sine-Gordon model. This is explained in details in [7] and [20]. We simply notice here that in the IR limit, all ϵ 's go to infinity so one finds a vanishing boundary entropy in (6.13). In the UV limit, it is easy to solve the system (6.11). One finds

$$\begin{aligned} 1 + x_k &= \frac{(k+1)^2}{k(k+2)} \\ 1 + x_a &= t, \end{aligned} \quad (6.20)$$

where $x_k = e^{-\epsilon_k}$. Now we have the two identities just proved by inspection

$$\begin{aligned} \sum_{k=1}^{t-2} \ln \frac{(k+1)^2}{k(k+2)} &= \ln \frac{2(t-1)}{t}, \\ \sum_{k=1}^{t-2} k \ln \frac{(k+1)^2}{k(k+2)} &= (t-1) \ln(t-1) - (t-2) \ln t. \end{aligned} \quad (6.21)$$

Using the fact that

$$\begin{aligned} \hat{u}_k(0) &= k \left(2 - \frac{2}{t} - \frac{2H}{t} \right), \\ \hat{u}_{t-1}(0) &= (t-1) \left(2 - \frac{2}{t} - \frac{2H}{t} \right), \\ \hat{u}_a(0) &= -2 + \frac{2}{t} + \frac{2H}{t}, \end{aligned} \quad (6.22)$$

one finds that

$$\sum_k \ln(1 + x_k) \int_{-\infty}^{\infty} u_k(\alpha) = 0. \quad (6.23)$$

Hence the boundary entropy is actually the same in the UV and IR limits, as was argued for a scalar theory (4.1).

7. The Destri De Vega equations for the boundary sine-Gordon model

7.1. The DDV equations with boundary conditions

We now would like to compute the complete Casimir effect in a theory with boundary. TBA in the R channel is pretty intricate in the non diagonal case. We use an alternative method elaborated by Destri and De Vega (DDV) in the periodic case. The lattice regularization plays a crucial role here - it is actually not known yet how the DDV method is related to the standard description of the theory using excitations and S matrices. Interestingly, the result are very close to those of the TBA of scalar theories in the R channel, and with some more work could probably be considered as a more rigorous proof of the results presented in the first part.

For that consider eq. (6.2) which we rewrite as

$$2Np(\alpha_j) + p_{bdry}(\alpha_j) + \sum_{\alpha_m > 0} \phi(\alpha_j - \alpha_m) + \phi(\alpha_j + \alpha_m) = 2\pi n_j, \quad (7.1)$$

where the sum runs over all roots (including $m = j$) and we introduced the notations

$$p(\alpha) \equiv \frac{1}{2} [f(\alpha + \Lambda, \gamma) + f(\alpha - \Lambda, \gamma)], \quad p_H(\alpha) \equiv f(\alpha, \gamma H), \quad \phi(\alpha) \equiv \phi_{11}(\alpha), \quad (7.2)$$

$$p_{bdry}(\alpha) = 2p_H(\alpha) - \phi(2\alpha).$$

The ground state is obtained by filling the real positive solutions, $\alpha_j = 0$ excepted. This corresponds to the choice $n_j = 1, 2, \dots$. Recall that if α_j is solution of (7.1) with some n_j , so is formally $-\alpha_j$ (with $-n_j$). Given the set of roots $\{\alpha_j > 0\}$ representing the ground state, one can construct the counting function as follows:

$$f(\alpha) \equiv 2iNp(\alpha) + ip_{bdry}(\alpha) + i \sum_{\alpha_m > 0} \phi(\alpha - \alpha_m) + \phi(\alpha + \alpha_m), \quad (7.3)$$

Define then

$$Y(\alpha) \equiv e^{f(\alpha)}. \quad (7.4)$$

We have $Y(\alpha_j) = Y(-\alpha_j) = 1$ for every root α_j from the ground state, as well as $Y(0) = 1$. Therefore we can rewrite (7.3) as

$$f(\alpha) = 2iNp(\alpha) + ip_{bdry}(\alpha) - i\phi(\alpha) - \int_C \phi(\alpha - \alpha') \frac{\dot{Y}(\alpha')}{1 - Y(\alpha')} \frac{d\alpha'}{2\pi}, \quad (7.5)$$

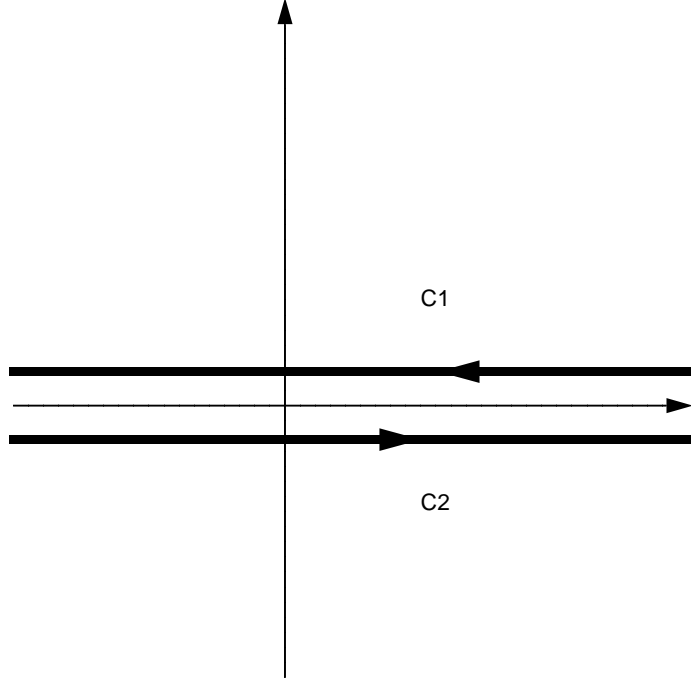


Figure 3. Contours of integration in the DDV method.

where $i\phi(\alpha)$ at the right-hand side takes care of the unwanted contribution of the pole $\alpha' = 0$ and the contour C consists of two parts as shown in figure 3, C_1 above and C_2 below the real axis.

Like in the bulk case [2], simple manipulations allow us to rewrite this non-linear integral equation as

$$f(\alpha) = 2iNP(\alpha) + iP_{bdry}(\alpha) + \int_{C_1} \Phi(\alpha - \alpha') \ln \left(1 - e^{f(\alpha')} \right) d\alpha' + \int_{C_2} \Phi(\alpha - \alpha') \ln \left(1 - e^{-f(\alpha')} \right) d\alpha'. \quad (7.6)$$

In equation (7.6) one has

$$\Phi(\alpha) = -\frac{t}{2\pi^2} \int_{-\infty}^{+\infty} dx e^{-itx\alpha/\pi} \frac{\sinh(t-2)x}{2 \sinh(t-1)x \cosh x}, \quad (7.7)$$

together with

$$P(\alpha) = \int_{-\infty}^{+\infty} dx \frac{e^{-itx\alpha/\pi} - 1}{-ix} \frac{\cos(\Lambda tx/\pi)}{2 \cosh x} \quad (7.8)$$

and

$$P_{bdry} = \int_{-\infty}^{+\infty} dx \frac{e^{-itx\alpha/\pi} - 1}{-ix} \left[\frac{\sinh(t-H)x}{\sinh(t-1)x \cosh x} + \frac{\sinh(t-2)x/2 \cosh tx/2}{\sinh(t-1)x \cosh x} + \frac{\sinh(t-2)x}{2 \sinh(t-1)x \cosh x} \right], \quad (7.9)$$

Obtaining (7.6) requires some care with the definition of logarithms. One proceeds as follows. Before integrating by parts in the integral over C_2 one factors out $\frac{d}{dx} \ln Y(x)$:

$$-\frac{\dot{Y}(x)}{1-Y(x)} = \frac{d}{dx} \ln[1-Y^{-1}] + \frac{d}{dx} \ln Y(x).$$

Then both integrals over C_1 and C_2 can be taken by parts, resulting in

$$\begin{aligned} f(\alpha) - \int_{-\infty}^{+\infty} \dot{\phi}(\alpha - \alpha') f(\alpha') \frac{d\alpha'}{2\pi} &= 2iNp(\alpha) + ip_{bdry}(\alpha) - i\phi(\alpha) \\ &- \int_{-\infty}^{+\infty} \dot{\phi}(\alpha - \alpha' - i0) \ln[1 - Y(\alpha' + i0)] \frac{d\alpha'}{2\pi} \\ &+ \int_{-\infty}^{+\infty} \dot{\phi}(\alpha - \alpha' + i0) \ln[1 - Y^{-1}(\alpha' - i0)] \frac{d\alpha'}{2\pi}, \end{aligned}$$

(surface terms from C_1 and C_2 cancel against each other provided N, Λ are finite). To make the source term vanish at infinity we take the derivative of both sides in the latter equation, after which it can be Fourier-transformed and “dressed” by the factor $(1 - \hat{\phi})^{-1}$. Finally, one goes back in the rapidity space and integrates the equation, using $f(0) = 0$, to obtain (7.6).

The energy of the ground state configuration can be expressed as

$$\begin{aligned} E^{latt} &= -\frac{2}{t} \sum_{\alpha_j > 0} \dot{p}(\alpha_j) = -\frac{1}{t} \sum_{\alpha_j} \dot{f}(\alpha_j - \Lambda, \gamma) + \dot{f}(-\alpha_j - \Lambda, \gamma) \\ &= \frac{1}{t} \dot{f}_\gamma(\Lambda) + \frac{1}{t} \int_C \dot{f}_\gamma(\alpha - \Lambda) \frac{\dot{Y}(\alpha)}{1 - Y(\alpha)} \frac{d\alpha}{2i\pi}, \end{aligned} \quad (7.10)$$

where $f(\alpha, \gamma) \equiv f_\gamma(\alpha)$. One finds after exactly the same manipulations as above,

$$\begin{aligned} E &= \frac{1}{t} \dot{f}_\gamma(\Lambda) - \frac{1}{t} \int_{-\infty}^{+\infty} f''_\gamma(\alpha - \Lambda + i0) \ln[1 - Y(\alpha + i0)] \frac{d\alpha}{2i\pi} \\ &+ \frac{1}{t} \int_{-\infty}^{+\infty} f''_\gamma(\alpha - \Lambda - i0) \ln[1 - Y^{-1}(\alpha - i0)] \frac{d\alpha}{2i\pi} + \frac{1}{t} \int_{-\infty}^{+\infty} f''_\gamma(\alpha - \Lambda) f(\alpha) \frac{d\alpha}{2i\pi}. \end{aligned} \quad (7.11)$$

Substituting (7.6) instead of $f(\alpha)$ in the last term of the latter, we obtain

$$\begin{aligned} E^{latt} = & E_{bulk}^{latt} + E_{bdry}^{latt} - \frac{i}{t} \int_{-\infty}^{+\infty} s(y - \Lambda + i0) \ln[1 - Y(y + i0)] \frac{dy}{2\pi} \\ & + \frac{i}{t} \int_{-\infty}^{+\infty} s(y - \Lambda - i0) \ln[1 - Y^{-1}(y - i0)] \frac{dy}{2\pi}, \end{aligned} \quad (7.12)$$

where we defined $s(y)$ by

$$s(y) = i \int_{-\infty}^{+\infty} \frac{k dk}{2 \cosh \gamma k} e^{-iky} = \frac{t^2 \tanh(ty/2)}{4 \cosh(ty/2)}. \quad (7.13)$$

The last two terms in (7.12) represent finite-size corrections to the ground state energy, while

$$\begin{aligned} E_{bulk}^{latt} + E_{bdry}^{latt} &= \frac{1}{t} \dot{f}_\gamma(\Lambda) - \frac{1}{t} \int_{-\infty}^{+\infty} \frac{dk}{2\pi} e^{-ik\Lambda} \hat{f}_\gamma(k) \frac{2N \hat{p}(k) + \hat{p}_{bdry}(k) - \hat{\phi}(k)}{2\pi - \hat{\phi}(k)} \\ &\equiv \frac{1}{t} \dot{f}_\gamma(\Lambda) - \frac{2N}{t} \int_{-\infty}^{+\infty} \dot{f}_\gamma(\alpha - \Lambda) \rho(\alpha) d\alpha, \end{aligned} \quad (7.14)$$

where the function $\rho(\alpha)$ defined so satisfies the following equation:

$$2\pi \rho(\alpha) = \dot{p}(\alpha) + \dot{\phi} * \rho(\alpha) + \frac{1}{2N} \left(\dot{p}_{bdry}(\alpha) - \dot{\phi}(\alpha) \right), \quad (7.15)$$

which can be checked by solving this linear equation in Fourier space. Introduce $\rho_1(\alpha) = \rho(\alpha) - \delta(\alpha)/2N$. Then

$$E_{bulk}^{latt} + E_{bdry}^{latt} = -\frac{2N}{t} \int_{-\infty}^{+\infty} \dot{f}_\gamma(\alpha - \Lambda) \rho_1(\alpha) d\alpha \quad (7.16)$$

and, by virtue of (7.15), ρ_1 satisfies the equation

$$2\pi \rho_1(\alpha) = \dot{p}(\alpha) + \dot{\phi} * \rho_1(\alpha) + \frac{1}{2N} \left(\dot{p}_{bdry}(\alpha) - 2\pi \delta(\alpha) \right). \quad (7.17)$$

Hence ρ_1 is the density of the ground state configuration (6.4) (see also (6.8)) and E_{bulk}^{latt} and E_{bdry}^{latt} coincide with the quantities computed in the previous section.

7.2. The continuum DDV equations

Having checked the correct values of bulk and boundary energies, we now let the cutoff $\Lambda \rightarrow \infty$ according to $\Lambda = \frac{2}{t} \log(2/m\Delta)$ with the size of the system $R = N\Delta$ and the physical mass $m = \frac{2}{\Delta} e^{-t\Lambda/2}$ fixed, and work only with the renormalized theory. In that limit one has, evaluating all integrals in Fourier transforms and keeping the leading terms,

$$s(\alpha + \Lambda) + s(\alpha - \Lambda) = -\frac{t^2}{2} m \sinh \theta, \quad NP(\theta) = mR \sinh \theta, \quad (7.18)$$

where we set

$$\theta = \frac{t\alpha}{2}. \quad (7.19)$$

Recall indeed that when one studies the excitations of the model, a relativistic dispersion relation is obtained provided the rapidity in the relativistic theory and the “bare rapidity” of the Bethe excitations are related by (7.19). We redefine implicitly all functions to depend on θ from now on. The energy therefore reads now,

$$\begin{aligned} E = & E_{bulk} + E_{bdry} + \frac{1}{2} \int_{C_1} m \sinh \theta \ln \left(1 - e^{f(\theta)} \right) \frac{d\theta}{2i\pi} \\ & + \frac{1}{2} \int_{C_2} m \sinh \theta \ln \left(1 - e^{-f(\theta)} \right) \frac{d\theta}{2i\pi}, \end{aligned} \quad (7.20)$$

where f is solution of the integral equation

$$\begin{aligned} f(\theta) = & 2imR \sinh \theta + iP_{bdry}(\theta) + \int_{C_1} \Phi(\theta - \theta') \ln \left(1 - e^{f(\theta')} \right) d\theta' \\ & + \int_{C_2} \Phi(\theta - \theta') \ln \left(1 - e^{-f(\theta')} \right) d\theta', \end{aligned} \quad (7.21)$$

and

$$\Phi(\theta) = - \int_{-\infty}^{\infty} \frac{dx}{2\pi^2} \frac{\sinh(t-2)x}{\cosh x \sinh(t-1)x} e^{2ix\theta/\pi}, \quad (7.22)$$

which can be identified with

$$\Phi(\theta) = -\frac{1}{2i\pi} \frac{d}{d\theta} \ln S_{++}(\theta), \quad (7.23)$$

where S_{++} is the soliton-soliton S matrix element. We have also from (7.9) and the identity (18) in [7] (see also [21])

$$P_{bdry}(\theta) = -\frac{1}{i} \ln S_{++}(2\theta) + \frac{2}{i} \ln \mathcal{R}_{++}(\theta) + \pi, \quad (7.24)$$

(so $\frac{d}{d\theta} P_{bdry} = \Theta + 2\pi\delta(\theta)$). The additional π in the above equation arises because the S_{++} has an overall minus sign [22] (eg $S = -1$ in the free fermion case $t = 2$).

7.3. The Casimir effect

We define the effective central charge by the formula

$$E = E_{bulk} + E_{bdry} - \frac{\pi c_{eff}}{24R}. \quad (7.25)$$

Observe that in the approach of the first part, we dealt with a renormalized theory with the vacuum energy subtracted. The last term in (7.25) is a renormalized energy, and it is what we will compare later to the results of the first part. From (7.20) we find

$$c_{eff}(mR) = -\frac{6R}{i\pi^2} \left\{ \int_{C_1} m \sinh \theta \ln \left(1 - e^{f(\theta)} \right) d\theta + \int_{C_2} m \sinh \theta \ln \left(1 - e^{-f(\theta)} \right) \right\} d\theta. \quad (7.26)$$

To study the ultraviolet behaviour of the above expression, let us use $f(\bar{\theta}) = \overline{f(\theta)}$ and rewrite (7.21) as

$$f(\theta) = 2imR \sinh \theta + iP_{bdry}(\theta) - 2i \int_{-\infty}^{\infty} \Phi(\theta - \theta') \operatorname{Im} \ln \left[1 - e^{f(\theta' + i0)} \right] d\theta', \quad (7.27)$$

and (7.26) as

$$c_{eff}(mR) = \frac{12mR}{\pi^2} \int_{-\infty}^{\infty} d\theta \sinh \theta \operatorname{Im} \ln \left[1 - e^{f(\theta + i0)} \right]. \quad (7.28)$$

It might be useful to remind the reader of the form of the corresponding bulk equations [2], [23]:

$$c_{eff}(mR) = \frac{6mR}{\pi^2} \int_{-\infty}^{\infty} \sinh \theta \operatorname{Im} \ln \left[1 + e^{f(\theta + i0)} \right] d\theta, \quad (7.29)$$

with f satisfying

$$f(\theta) = imR \sinh \theta + i\omega - 2i \int_{-\infty}^{\infty} \Phi(\theta - \theta') \operatorname{Im} \ln \left[1 + e^{f(\theta' + i0)} \right] d\theta', \quad (7.30)$$

and ω is the twist of the 6-vertex model. Note the deep similarity between these two systems; the factors of 2 can obviously be absorbed in a redefinition of mR and the minus sign in the arguments of logarithms in a redefinition of f , so the only essential difference is that the twist angle ω is replaced by P_{bdry} . It is well known that the twist corresponds to a soliton fugacity [7] so we recover the result of the first part that the boundary acts by some effective, rapidity dependent fugacity.

In the limit when $R \rightarrow 0$, only the region $|\theta|$ large contributes to c_{eff} . Let us focus on the limit $\theta \gg 1$, the results for negative θ following by symmetry. Then one finds $f \approx f_K$, where

$$f_K(\theta) = imRe^\theta + iP_{bdry}(\infty) - 2i \int_{-\infty}^{\infty} \Phi(\theta - \theta') \operatorname{Im} \ln \left[1 - e^{f_K(\theta' + i0)} \right] d\theta', \quad (7.31)$$

together with

$$c_K(mR) = \frac{6}{\pi^2} \int_{-\infty}^{\infty} mRe^{\theta} \operatorname{Im} \ln \left[1 - e^{f_K(\theta+i0)} \right] d\theta. \quad (7.32)$$

It is now useful to recall some well known results about dilogarithms [24]. Define

$$L(x) \equiv \int_0^x du \left[\frac{\ln(1+u)}{u} - \frac{\ln u}{1+u} \right]. \quad (7.33)$$

Assume

$$-i \ln F(x) = \phi(x) + 2 \int_{-\infty}^{\infty} dy G(x-y) \operatorname{Im} \ln [1 + F(y+i0)], \quad (7.34)$$

with G an even function. Then one has

$$\begin{aligned} \operatorname{Im} \int_{-\infty}^{\infty} dx \phi'(x) \ln [1 + F(x+i\epsilon)] &= \frac{1}{2} \operatorname{Re} \{ L[F(-\infty)] - L[F(\infty)] \} \\ &+ \frac{1}{2} \operatorname{Im} \{ \phi(\infty) \ln [1 + F(\infty)] - \phi(-\infty) \ln [1 + F(-\infty)] \}, \end{aligned} \quad (7.35)$$

(where we did not write the $i0$ part of some arguments for simplicity). Set $F = e^{f_K - i\pi}$ and denote $P_{bdry}(\infty) \equiv \sigma$. Then, according to (7.31): $\phi = mRe^{\theta} + \sigma - \pi$, $G = -\Phi$. We have $\phi(-\infty) = \sigma - \pi$, $\phi(\infty) = \infty$. One has also $F(\infty+i0) = 0$, and from this and (7.31) one can get the value of F at $-\infty$:

$$\begin{aligned} \frac{f_K(-\infty)}{i} &= \sigma - 2 \operatorname{Im} \ln \left[1 - e^{f_K(-\infty+i0)} \right] \int_{-\infty}^{\infty} \Phi(\theta) d\theta \\ &= \sigma + \frac{t-2}{t-1} \operatorname{Im} \ln \left[1 - e^{f_K(-\infty+i0)} \right]. \end{aligned} \quad (7.36)$$

So, if $e^{f_K(-\infty+i0)} = e^{i\omega}$ one may use $\operatorname{Im} \ln(1 \pm e^{i\omega}) = \frac{1}{2i} \ln(\pm e^{i\omega})$ to find

$$e^{i\omega} = -\exp \left\{ 2i \frac{t-1}{t} \sigma + 2i \frac{\pi}{t} \right\}. \quad (7.37)$$

From (7.35) it follows that

$$\operatorname{Im} \int_{-\infty}^{\infty} mRe^{\theta} \ln(1 - e^{f_K(\theta+i0)}) d\theta = \frac{1}{2} \operatorname{Re} L(-e^{i\omega}) - \frac{1}{2} (\sigma - \pi) \left(\frac{t-1}{t} \sigma - \pi + \frac{\pi}{t} \right).$$

In the region $\theta \ll 1$ we have $f_1 \approx f_A$, and similar calculations yield

$$\operatorname{Im} \int_{-\infty}^{\infty} mRe^{-\theta} \ln(1 - e^{f_A(\theta+i0)}) d\theta = -\frac{1}{2} \operatorname{Re} L(-e^{-i\omega}) + \frac{1}{2} (\pi - \sigma) \left(-\frac{t-1}{t} \sigma + \pi - \frac{\pi}{t} \right).$$

Collecting both $\theta \gg 1$ and $\theta \ll 1$ contributions we obtain

$$\begin{aligned} c_{UV} &= \frac{6}{\pi^2} \left\{ \frac{1}{2} \left[L(-e^{2i\omega}) + L(-e^{-2i\omega}) \right] + \frac{t-1}{t} (\sigma - \pi)^2 \right\} = \\ &= \frac{6}{\pi^2} \left[\frac{\pi^2}{6} - \frac{t-1}{t} (\sigma - \pi)^2 \right]. \end{aligned} \quad (7.38)$$

From (7.9) we get

$$P_{bdry}(\infty) \equiv \sigma = 2\pi - \frac{\pi}{2} \frac{H + H'}{t - 1}. \quad (7.39)$$

Finally, from this we find

$$c_{UV} = 1 - 6 \frac{t-1}{t} \left(1 - \frac{H + H'}{2(t-1)} \right)^2. \quad (7.40)$$

In the case with no boundary field, $H = H' = t - 1$ so $c_{UV} = 1$ as expected.

Remark. So far we tacitly assumed that $0 < H < t - 1$. In general, from relation (6.3) follows that when $h > 0$, H varies between $-t - 1$ and -1 , while when $h < 0$, $-1 < H < t - 1$. To generalize our results, we should use the most general form of p_H :

$$\hat{p}_H(k) = \int_{-\infty}^{\infty} \dot{p}_H(\alpha) e^{ik\alpha} d\alpha = 2\pi \operatorname{sign}(H) \frac{\sinh(\pi - \omega_H)k}{\sinh \pi k}, \quad -\pi < \gamma H < \pi, \quad (7.41)$$

where we defined $\omega_H \equiv |\gamma H|$. For $-2\pi < \gamma H < -\pi$ set $\omega_H = 2\pi + \gamma H$ and $\operatorname{sign}(H) = 1$ in (7.41). Then (7.39) generalizes to

$$\sigma = 2\pi - \frac{\pi}{2} \frac{\omega_H + \omega'_H}{\pi - \gamma}$$

if H and H' are both positive or $-2t < (H, H') < -t$, and

$$\sigma = 2\pi - \frac{\pi}{2} \frac{4\pi - \omega_H - \omega'_H}{\pi - \gamma}$$

if they are both negative, but greater than $-t$. In the case when $0 < H < t - 1$ and $-t < H' < -1$ (that is, $h < 0, h' > 0$) we get:

$$\sigma = \pi + \frac{\pi}{2} \frac{\omega'_H - \omega_H - 2\gamma}{\pi - \gamma}.$$

So, when $\omega'_H - \omega_H = 2\gamma$ we have $\sigma = \pi$ and $c_{UV} = 1$, as in the free case. The condition $\omega'_H - \omega_H = 2\gamma$ is equivalent to $h = -h'$, as could be seen from (6.3). That $c = 1$ when the two surface field are real and opposite is well known from lattice studies [25].

8. Remarks on the DDV equations for minimal models and excited states

A particularly interesting case is when the XXZ chain or the inhomogeneous 6-vertex model commutes with the quantum group $U_q sl(2)$. In that case $h = -h' = 2i \sin \gamma$ and the net result is that all H dependent terms simply disappear from the equations so in particular

$$E_{bdry}^{qu} = \frac{m}{2} \left(\cot \frac{t\pi}{4} + 1 \right). \quad (8.1)$$

At the $N = 2$ supersymmetric point, $t = 3$ so the boundary energy vanishes, a result well expected from supersymmetric considerations. More generally, it vanishes if $t = 4n + 3$, n an integer. Notice that the bulk energy vanishes for t an odd number (as a consequence of the generalized fractional $N = 2$ supersymmetries studied in [26]).

In the quantum group symmetric case [27] one has $H + H' = 2t$ so from (7.40)

$$c = 1 - \frac{6}{t(t-1)}, \quad (8.2)$$

the expected result for the restricted sine-Gordon model[6][28][29].

In the quantum group case, all H -dependent terms simply disappear from the equations (in notations of [3] this corresponds to $\xi \rightarrow \pm i\infty$). We find then the value of the boundary matrix element

$$\ln \mathcal{R} = \frac{1}{i} \int_{-\infty}^{\infty} \frac{dx}{x} \frac{\sinh(3x/2) \sinh(t-2)x/2}{\sinh(t-1)x/2 \sinh(t-1)x} \sin \frac{2}{\pi} x\theta. \quad (8.3)$$

In the last equation we suppressed the label $++$ because the scattering of solitons and antisolitons can be made identical by the same change of gauge which ensures commutation of the bulk S-matrix with $U_q sl(2)$. To understand (8.3), we can use the reformulation of the quantum group symmetric bulk-scattering in terms of the $O(n)$ model. In that case, the bulk S-matrix can be rewritten [30]

$$S_{ab}^{cd} = \frac{\rho(-i\theta)}{i} \left[\sinh \lambda(i\pi - \theta) \delta_a^b \delta_c^d + \sinh \lambda\theta \delta_{ac} \delta^{bd} \right], \quad (8.4)$$

where

$$\rho(-i\theta) = -i \sinh^{-1} \lambda(i\pi - \theta) \exp \left[\frac{1}{i} \int_0^{\infty} \frac{dx}{x} \frac{\sinh \left(\left(\frac{1}{\lambda} - 1 \right) x \right)}{\sinh \left(\frac{x}{\lambda} \right) \cosh(x)} \sin \frac{2}{\pi} x\theta \right]. \quad (8.5)$$

The labels correspond to colors in an $O(n)$ symmetric model, running formally $a = 1, \dots, n$. One has $n = 2 \cosh i\pi\lambda$ so the sine-Gordon model corresponds to $n < 2$ and one must

continue things analytically in n . In our previous notations, $\lambda = \frac{1}{t-1}$. In this approach, the particles are self conjugate so the unitarity crossing relation reads

$$\mathcal{R}_a^a\left(i\frac{\pi}{2} - \theta\right) = S_{aa}^{bb}(2\theta)\mathcal{R}_b^b\left(i\frac{\pi}{2} + \theta\right). \quad (8.6)$$

We deal with the $O(n)$ model with free boundary conditions so by $O(n)$ symmetry, \mathcal{R}_a^a actually does not depend on a let us just call it \mathcal{R} . Then (8.6) reads

$$\begin{aligned} \mathcal{R}\left(i\frac{\pi}{2} - \theta\right) &= \frac{\rho(-2i\theta)}{i} [n \sinh 2\lambda\theta + \sinh \lambda(i\pi - 2\theta)] \mathcal{R}\left(i\frac{\pi}{2} + \theta\right) \\ &= \sin \lambda(\pi - 2i\theta) \rho(-2i\theta) \mathcal{R}\left(i\frac{\pi}{2} + \theta\right), \end{aligned} \quad (8.7)$$

where we used $n = 2 \cosh i\pi\lambda$. The latter equation appear in [3], and its minimal solution is precisely (8.3). More results about the boundary $O(n)$ model will be presented elsewhere, together with a kink interpretation for minimal models.

Now in [25], it was shown how one can obtain the various conformal weights h_{rs} by changing the boundary conditions and the total value of the spin in the XXZ chain. The change of boundary conditions proposed in [25] was to project the first r spins near a boundary onto the spin r representation of $U_q sl(2)$, which leads to characters $h_{r,1}$. Now let us recall (see [31] for discussion, see also [32]) that if $\mathcal{R}_{++}, \mathcal{R}_{--}$ are boundary Reflection-matrices satisfying unitarity and crossing, one can obtain new solutions of the same equations by defining

$$\begin{aligned} \mathcal{R}_{++}^{(2)}(\theta) &\equiv a(\theta - \theta_0)a(\theta + \theta_0)\mathcal{R}_{++}(\theta) \\ \mathcal{R}_{--}^{(2)}(\theta) &\equiv b(\theta - \theta_0)b(\theta + \theta_0)\mathcal{R}_{--}(\theta) + c(\theta - \theta_0)c(\theta + \theta_0)\mathcal{R}_{++}(\theta), \end{aligned} \quad (8.8)$$

where a, b, c denote the standard elements of the bulk sine-Gordon S-matrix, and θ_0 is a free parameter. The process can be iterated again to give R-matrices $\mathcal{R}^{(n)}$. Now one can again let $\xi \rightarrow \pm i\infty$ to obtain what we conjecture to be the continuum limit of the projection discussed in [25]. The DDV equations read as above, but with $\mathcal{R}^{(r)}$ replacing \mathcal{R} . The effective central charge follows from $P_{bdry}(\infty)$. Using

$$\frac{2}{i} \ln S_{++}(\infty) = -\pi \frac{t}{t-1}$$

one finds

$$P_{bdry}(\infty) = \pi \left(\frac{t-2}{t-1} - (r-1) \frac{t}{t-1} \right). \quad (8.9)$$

Using the equations (7.36) to (8.2) this leads to

$$c_{eff}^{(r)} = 1 - 6 \frac{t-1}{t} \left(1 - \frac{t}{t-1} r \right)^2 = c - 24h_{r1}. \quad (8.10)$$

Hence we believe we can observe the conformal weights h_{r1} using these “boundary fused” R-matrices (what value to give to θ_0 requires more discussion).

Changing the value of s is more difficult from the DDV point of view. On the lattice, all one has to do is to look at non vanishing spin m . Then one observes $h_{r,1+2m}$, corresponding to

$$P_{bdry}(\infty) = \pi \left(-\frac{1}{t-1} - (r-1) \frac{t}{t-1} + (s-1)\pi \right). \quad (8.11)$$

Unfortunately all our equations are defined modulo 2π . So it is possible to observe $h_{r,2}$ only. For that purpose simply subtract π to (7.24) (this corresponds to having a lattice with an odd number of sites).

9. The massless DDV equations

The structure of the DDV equations is pretty transparent in continuum terms: the function Φ takes care of the interaction between solitons and antisolitons, and the term P_{bdry} takes care of all the boundary effects. It is then natural to guess what the DDV equations would look like in other cases. A case of special interest is the bulk massless case. If the boundary is also conformal invariant, one gets the set of equations (7.31) and (7.32), and we see that the particular conformal boundary condition is fully characterized by **a rapidity independent phase**. If the boundary is perturbed and the perturbation is integrable, its action can be described by a boundary scattering. If moreover this scattering is diagonal and identical for solitons and antisolitons, it is easy to write the equations as

$$\begin{aligned} f(\theta) &= imRe^\theta + iP_{bdry}(\theta - \theta_B) - 2i \int_{-\infty}^{\infty} \Phi(\theta - \theta') \operatorname{Im} \ln \left[1 - e^{f(\theta' + i0)} \right] d\theta' \\ c(me^{\theta_B} R) &= \frac{12}{\pi^2} \int_{-\infty}^{\infty} mRe^\theta \operatorname{Im} \ln \left[1 - e^{f(\theta + i0)} \right] d\theta. \end{aligned} \quad (9.1)$$

Here θ_B characterizes the strength of the boundary perturbation, and me^{θ_B} is the usual “Kondo” or “boundary” temperature. The only θ dependent term in P_{bdry} is the one involving the boundary S-matrix. The simplest case corresponds to the following boundary S-matrix

$$\mathcal{R}(\theta - \theta_B) = i \tanh \left(\frac{\theta - \theta_B}{2} - \frac{i\pi}{4} \right), \quad (9.2)$$

corresponding to an interaction of anisotropic Kondo type on one side of the strip, and we chose the usual Dirichlet boundary conditions on the other side. Then one has

$$P_{bdry}(\theta - \theta_B) = \frac{1}{i} \ln \tanh \left(\frac{\theta - \theta_B}{2} - \frac{i\pi}{4} \right). \quad (9.3)$$

As a result, the energy scales as the ground state of the free boson theory with $c = 1$ in the UV, but according to (7.40) in the IR one observes an excited state of conformal weight

$$h = \frac{t-1}{4t} \left(\frac{P_{bdry}(\infty)}{\pi} \right)^2 = \frac{t-1}{4t}. \quad (9.4)$$

Microscopically, the IR limit corresponds to having an infinite boundary field in the XXZ chain and one easily checks directly the validity of (9.4).

10. Conclusions

To conclude, consider the energy of the renormalized theory, $E = -\frac{\pi c_{eff}}{24R}$. So far, the relation between the DDV equations and the underlying scattering structure is unknown. Nevertheless, it is likely that the two terms (C_1 and C_2) in the above equations correspond to the existence of “soliton” and “antisoliton”. By naive extension, the result for a scalar theory should have the same form but with only one of these terms. After shifting the contour C_1 to $\text{Im}(\theta) = -i\pi/2$ and changing a few notations, one is thus led to the following conjecture for the ground state energy of a **scalar** theory with bulk scattering given by Φ :

$$E = -\frac{m}{4\pi} \int_{-\infty}^{\infty} \cosh \theta \ln \left[1 + e^{-\epsilon(\theta)} \right] d\theta, \quad (10.1)$$

where now

$$\epsilon(\theta) = 2mR \cosh \theta - iP_{bdry}(\theta + i\pi/2) + i\pi + \int_{-\infty}^{\infty} \Phi(\theta - \theta') \text{Im} \ln \left[1 + e^{-\epsilon(\theta')} \right] d\theta', \quad (10.2)$$

and

$$\Phi(\theta) = -\frac{1}{2i\pi} \frac{d}{d\theta} \ln S(\theta). \quad (10.3)$$

Observe that, introducing the notation $K(\theta) = \mathcal{R} \left(\frac{i\pi}{2} - \theta \right)$ we can rewrite

$$i\pi - iP_{bdry}(\theta + i\pi/2) = -\ln K(-\theta)K'(-\theta) - \ln S(2\theta), \quad (10.4)$$

where we used the relations $S(\theta)S(-\theta) = 1$ and $S(i\pi - \theta) = S(\theta)$. We can also use the boundary crossing relation [3]

$$K(\theta) = S(2\theta)K(-\theta), \quad (10.5)$$

together with $\overline{K}(\theta) = K(-\theta)$ to rewrite

$$-iP_{bdry}(\theta + i\pi/2) + i\pi = -\ln \overline{K}(\theta) - \ln K'(\theta). \quad (10.6)$$

This is exactly the equation we deduced in section 3.

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